Fachbereich C – Mathematik und Naturwissenschaften
– Physik –

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Theoretical Solid State Physics, WS 08/09

13th practice sheet

Closing date: 29.01.2009, at 1:00 pm into the PO Box

28. **BCS gap equation I (6 points)**

The BCS Hamiltonian was introduced in the lecture. Applying a mean-field approximation $AB \to A\langle B\rangle + B\langle A\rangle - \langle A\rangle\langle B\rangle$ for the annihilation and creation operators of cooper pairs A and B yields:

$$H_{\text{eff}} = \sum_{k,\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} - \Delta \sum_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} - \Delta^* \sum_k c_{k\uparrow} c_{-k\downarrow} + \frac{V|\Delta|^2}{g} \tag{1}$$

with $\Delta := g\sum_k \langle c_{k\uparrow} c_{-k\downarrow}\rangle$. For real Δ the Hamiltonian H_{eff} can be diagonalized by the Bogoliubov transformation $\alpha_k := u_k c_{k\uparrow} - v_k c^\dagger_{-k\downarrow}$, $\beta_k := u^*_k c^\dagger_{-k\downarrow} + v^*_k c_{k\uparrow}$ (cf. problem 25):

$$H_{\text{eff}} = \sum_k \Big(E_k(\alpha^\dagger_k \alpha_k - \beta^\dagger_k \beta_k) + \epsilon_k \Big) + \frac{V|\Delta|^2}{g}, \qquad E_k := \sqrt{\epsilon_k^2 + \Delta^2}. \tag{2}$$

Calculate $\langle c_{k\uparrow} c_{-k\downarrow}\rangle_{H_{\text{eff}}}$ by introducing the inverse Bogoliubov transformation and derive the so-called *BCS gap equation*:

$$\Delta = \frac{\Delta g}{2} \sum_k \frac{\tanh(\beta E_k/2)}{E_k}. \tag{3}$$

29. **BCS gap equation II (6 points)**

We want to analyse the *BCS gap equation* for the superconducting order parameter Δ. It is zero in the normal and non-zero in the superconducting state. The cause of cooper pairing is the attractive interaction of electrons, which can be understood within the theory of electron-phonon coupling. Therefore the sum over all k's can be substituted by an integration over all states with energies between $-\hbar\omega_D$ and $\hbar\omega_D$ (only coupling to acoustic phonons):

$$1 = \frac{g\rho_0}{2} \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{\tanh\left(\frac{\beta}{2}\sqrt{\epsilon^2+\Delta^2}\right)}{\sqrt{\epsilon^2+\Delta^2}} d\epsilon, \tag{4}$$

where we have assumed that the density of states $\rho(\epsilon)$ is constant in the intervall $[-\hbar\omega_D, \hbar\omega_D]$.

(a) Calculate $\Delta_0 := \lim\limits_{T\to 0} \Delta(T)$.

(b) Analyse the integral in equation (4) for high temperatures. What is the solution of the BCS gap equation (3)?

(c) The condition $\Delta(T_c) = 0$ in equation (4) fixes the critical temperature. Calculate T_c! First of all integrate by parts, then assume $\hbar\omega_D \gg k_B T_c$ and use $\int_0^\infty \ln x(1-\tanh^2 x)dx = \ln(\pi/4) - \gamma$, $\gamma = 0,5772\ldots$.