Fachbereich C – Mathematik und Naturwissenschaften
– Physik –

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Theoretical Solid State Physics, WS 08/09

12th practice sheet

Closing date: 22.01.2009, at 1:00 pm into the PO Box

26. **BCS groundstate (10 points)**

Within the BCS-theory the groundstate of a superconductor can be desribed by the α- and β-particles (cf. sheet 11 and the lecture). The β-band is completely filled up while the α-band is empty. Thus the groundstate $|\Psi_0\rangle$ can be defined as follows:

$$\langle\Psi_0|\Psi_0\rangle = 1 \qquad \text{and} \qquad \alpha_k|\Psi_0\rangle = \beta_k^\dagger|\Psi_0\rangle = 0 \quad \forall\, k. \tag{1}$$

(a) Show, that $|\Psi_0\rangle$ is not an eigenstate of the operator of particle number $\hat{N} := \sum_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma}$.

Hint: Calculate $[\hat{N}, \beta_k^\dagger]$ using the relation $[AB, C] = A\{B, C\} - \{A, C\}B$. Then assume that $|\Psi_0\rangle$ is an eigenstate of $\hat{N}$, i.e. $\hat{N}|\Psi_0\rangle = N|\Psi_0\rangle$, and calculate $[\hat{N}, \beta_k^\dagger]|\Psi_0\rangle$ in two different ways.

(b) Let P_N be the projector which projects on the subspace with particle number N:

$$P_N := \frac{1}{2\pi}\int_0^{2\pi} e^{i(\hat{N}-N)\phi}\, d\phi \tag{2}$$

Show that

$$P_N = \frac{1}{2\pi}\int_0^{2\pi} e^{-iN\phi} \prod_{k\sigma}\left(1 + (e^{i\phi} - 1)c_{k\sigma}^\dagger c_{k\sigma}\right), \tag{3}$$

simplify this expression and prove the properties of projection, i.e. $P_N^2 = P_N$.

(c) Calculate $\langle\Psi_0|P_N|\Psi_0\rangle$.

27. **Wick's theorem (10 points)**

Consider non-interacting fermions described by a Hamiltonian of the form $H = \sum_\alpha \epsilon_\alpha c_\alpha^\dagger c_\alpha$.

(a) First of all show the identity:

$$c_\alpha^\dagger e^{-\beta(H-\mu N)} = e^{\beta(\epsilon_\alpha-\mu)} e^{-\beta(H-\mu N)} c_\alpha^\dagger, \tag{4}$$

where $N = \sum_\alpha c_\alpha^\dagger c_\alpha$ is the operator of particle number.

(b) Calculate the expectation value

$$\langle c_{\alpha_1}^\dagger c_{\alpha_2}\rangle = \frac{\text{tr } e^{-\beta(H-\mu N)} c_{\alpha_1}^\dagger c_{\alpha_2}}{\text{tr } e^{-\beta(H-\mu N)}}. \tag{5}$$

Hint: Put the operator $c_{\alpha_1}^\dagger$ to the right by using the appropriate commutation relations. Use the cyclic commutation under the trace and the result of part (a).

(c) Show the following formula for the (grand canonical) expectation value of $c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger c_{\alpha_3} c_{\alpha_4}$:

$$\langle c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger c_{\alpha_3} c_{\alpha_4}\rangle = \langle c_{\alpha_1}^\dagger c_{\alpha_4}\rangle\langle c_{\alpha_2}^\dagger c_{\alpha_3}\rangle - \langle c_{\alpha_1}^\dagger c_{\alpha_3}\rangle\langle c_{\alpha_2}^\dagger c_{\alpha_4}\rangle \tag{6}$$

(d) What are the modifications of this formula for bosons?