Fachbereich C – Mathematik und Naturwissenschaften
– Physik –

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Theoretical Solid State Physics, WS 08/09

11th practice sheet

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24. **Ginzburg-Landau theory (10 points)**

Ginzburg and Landau postulated the existence of a wave function $\Psi(\vec{r})$ describing the phenomenon of superconductivity:

$$|\Psi(\vec{r})|^2 = n_s(\vec{r}), \tag{1}$$

where n_s is the density of superconducting particles (mass m^*, charge e^*). For the enthalpy density they made the ansatz:

$$g_s(T,H) = g_n(T,0) + a|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e^*}{c}\vec{A}(\vec{r})\right)\Psi(\vec{r})\right|^2 + \frac{B^2}{8\pi} - \frac{\vec{B}\cdot\vec{H}}{4\pi}. \tag{2}$$

$g_n(T,0)$ is the enthalpy density in the normal state. $\vec{B} = \vec{\nabla}\times\vec{A}$ is the magnetic field in the superconductor. The enthalpy depends on the temperature T and the external magnetic field H. The condition that the enthalpy $G = \int_V d^3r g_s(\vec{r})$ is minimal leads to the thermodynamics of the superconducting state.

(a) First of all consider the case $\vec{A} = 0$, $\Psi(\vec{r}) = \text{const.} \neq 0$ and calculate a and b in terms of the particle density n_s and the critical field H_c.

(b) In order to analyse the effects of surfaces we readmit spatial variations of $\Psi(\vec{r})$ and $\vec{A}(\vec{r}) \neq 0$. From the variation of G respect to $\Psi^*(\vec{r})$ and $\vec{A}(\vec{r})$ it follows

$$0 = \frac{1}{2m^*}\left(\frac{\hbar}{i} - \frac{e^*}{c}\vec{A}(\vec{r})\right)^2 \Psi(\vec{r}) + a\Psi(\vec{r}) + b|\Psi(\vec{r})|^2\Psi(\vec{r}) \tag{3}$$

$$j_s := \frac{1}{4\pi}\text{rot}(\vec{B}-\vec{H}) = \frac{e^*\hbar}{2m^*i}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right) - \frac{e^{*2}}{m^*c}|\Psi|^2\vec{A}. \tag{4}$$

Prove these equations.
Hint: The first step is an integration by parts to eliminate terms with $\nabla\Psi^*$. Note that surface terms vanish upon local variations. For the proof of the second equation use the identity $\text{div}\,(\vec{a}\times\vec{b}) = (\text{rot}\,\vec{a})\cdot\vec{b} - \vec{a}\cdot(\text{rot}\,\vec{b})$ for any $\vec{a}$, $\vec{b}$ and Gauss's theorem.

(c) Consider a superconductor in the half space $x > 0$ and a normal conductor in the other half space $x < 0$. The equation (3) is reduced to a one-dimensional differential equation. Set $f(x) := \Psi(x)/|\Psi(\infty)|$ and solve the differential equation with the ansatz $f(x) := A\tanh(\alpha(x - x_0))$. Calculate A, α and x_0.

25. **Bogoliubov transformation (10 points)**

Let H be a non-diagonal Hamiltonian (in momentum representation):

$$H = \sum_{k,\sigma}\epsilon_k c^\dagger_{k\sigma}c_{k\sigma} - \Delta\sum_k\left[c_{-k\downarrow}c_{k\uparrow} + c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}\right] + \frac{V\Delta^2}{g}, \tag{5}$$

where Δ and g are constants and $c_{k\sigma}$, $c^\dagger_{k\sigma}$, $\sigma = \uparrow, \downarrow$ are fermionic creation and annihilation operators: $\{c_{k\sigma}, c^\dagger_{k'\sigma'}\} = \delta_{kk'}\delta_{\sigma\sigma'}$. The symbol $\{\cdot,\cdot\}$ is the anticommutator. The goal is to diagonalize this Hamiltonian.

(a) Show that the following operators

$$\alpha_k := u_k c_{k\uparrow} - v_k c^\dagger_{-k\downarrow}, \qquad \alpha^\dagger_k := u^*_k c^\dagger_{k\uparrow} - v^*_k c_{-k\downarrow}, \tag{6}$$
$$\beta_k := u^*_k c^\dagger_{-k\downarrow} + v^*_k c_{k\uparrow}, \qquad \beta^\dagger_k := u_k c_{-k\downarrow} + v_k c^\dagger_{k\uparrow}, \tag{7}$$

fulfil the relations

$$\{\alpha_k, \alpha^\dagger_{k'}\} = \{\beta_k, \beta^\dagger_{k'}\} = (|u_k|^2 + |v_k|^2)\delta_{kk'}, \tag{8}$$
$$\{\alpha_k, \alpha_{k'}\} = \{\beta_k, \beta_{k'}\} = \{\alpha_k, \beta_{k'}\} = \{\alpha_k, \beta^\dagger_{k'}\} = 0. \tag{9}$$

(b) Which condition do u_k and v_k have to satisfy, so that α_k and β_k are fermionic annihilators?

(c) Express the four c-operators by the α- and β-operators.

(d) Let u_k and v_k be real. Write the Hamiltonian H in terms of α- and β-operators.

(e) Which second condition do u_k and v_k have to satisfy, so that the Hamiltonian H is diagonal.

(f) Use the two conditions from part (b) and (e) to calculate u_k and v_k and write down the Hamiltonian as simply as possible.

The transformation in part (a) is called Bogoliubov transformation for fermions.