Exercise 13 for Theoretical Solid State Physics in Summer 2023

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Submission: 12.07.2023, 12:00 in the P.O. Box Popkov on D.10 (by e-mail)
Discussion: 12.07.2023, 14:15

1. **Wick's theorem (10 points)**
   Consider non-interacting fermions described by a Hamiltonian of the form $\mathbf{H} = \sum_\alpha \epsilon_\alpha \mathbf{c}^\dagger_\alpha \mathbf{c}_\alpha$.

   (a) First of all show the identity

   $$\mathbf{c}^\dagger_\alpha e^{-\beta(\mathbf{H}-\mu\mathbf{N})} = e^{\beta(\epsilon_\alpha-\mu)} e^{-\beta(\mathbf{H}-\mu\mathbf{N})}\mathbf{c}^\dagger_\alpha,$$

   where $\mathbf{N} = \sum_\alpha \mathbf{c}^\dagger_\alpha \mathbf{c}_\alpha$ is the operator of particle number.

   (b) Calculate the expectation value

   $$\left\langle \mathbf{c}^\dagger_\alpha \mathbf{c}_\beta \right\rangle = \frac{\mathrm{tr}\left(e^{-\beta(\mathbf{H}-\mu\mathbf{N})}\mathbf{c}^\dagger_\alpha \mathbf{c}_\beta\right)}{\mathrm{tr} e^{-\beta(\mathbf{H}-\mu\mathbf{N})}}$$

   Hint: Put the operator $\mathbf{c}^\dagger_\alpha$ to the right by using the appropriate commutation relations. Use the cyclic commutation under the trace and the result of part (a).

   (c) Show the following formula for the (grand canonical) expectation value of $\mathbf{c}^\dagger_{\alpha_1}\mathbf{c}^\dagger_{\alpha_2}\mathbf{c}_{\alpha_3}\mathbf{c}_{\alpha_4}$

   $$\left\langle \mathbf{c}^\dagger_{\alpha_1}\mathbf{c}^\dagger_{\alpha_2}\mathbf{c}_{\alpha_3}\mathbf{c}_{\alpha_4} \right\rangle = \left\langle \mathbf{c}^\dagger_{\alpha_1}\mathbf{c}_{\alpha_4}\right\rangle \left\langle \mathbf{c}^\dagger_{\alpha_2}\mathbf{c}_{\alpha_3}\right\rangle - \left\langle \mathbf{c}^\dagger_{\alpha_1}\mathbf{c}_{\alpha_3}\right\rangle \left\langle \mathbf{c}^\dagger_{\alpha_2}\mathbf{c}_{\alpha_4}\right\rangle$$

   (d) What are the modifications of this formula for bosons?

2. **Bogoliubov transformation (5 points)**
   Let $\mathbf{H}$ be a non-diagonal Hamiltonian (in momentum representation):

   $$\mathbf{H} = \sum_{k,\sigma} \epsilon_k \mathbf{c}^\dagger_{k,\sigma}\mathbf{c}_{k,\sigma} - \triangle \sum_k \left(\mathbf{c}_{-k,\downarrow}\mathbf{c}_{k,\uparrow} + \mathbf{c}^\dagger_{k,\uparrow}\mathbf{c}^\dagger_{-k,\downarrow}\right) + \frac{V\triangle^2}{g},$$

where $\triangle$ and g are constants and $\mathbf{c}^\dagger_{k,\sigma}$, $\mathbf{c}_{k,\sigma}$, $\sigma =\uparrow,\downarrow$, are fermionic creation and annihilation operators:

$$\left\{\mathbf{c}_{k,\sigma}, \mathbf{c}^\dagger_{k',\sigma'}\right\} = \delta_{kk'}\delta_{\sigma\sigma'}, \qquad \left\{\mathbf{c}^\dagger_{k,\sigma}, \mathbf{c}^\dagger_{k',\sigma'}\right\} = \{\mathbf{c}_{k,\sigma}, \mathbf{c}_{k',\sigma'}\} = 0$$

The symbol $\{\cdot,\cdot\}$ is the anticommutator:

$$\{\mathbf{A},\mathbf{B}\} = \mathbf{AB} + \mathbf{BA}$$

The goal is to diagonalize this Hamiltonian. Here we take a few (not all) steps into this direction.

(a) Show that the following operators

$$\begin{aligned}\boldsymbol{\alpha}_k &= u_k \mathbf{c}_{k,\uparrow} - v_k \mathbf{c}^\dagger_{-k,\downarrow},\\ \boldsymbol{\beta}_k &= u_k^* \mathbf{c}^\dagger_{-k,\downarrow} + v_k^* \mathbf{c}_{k,\uparrow}\end{aligned}$$

fulfil the relations

$$\begin{aligned}&\left\{\boldsymbol{\alpha}_k, \boldsymbol{\alpha}^\dagger_{k'}\right\} = \left\{\boldsymbol{\beta}_k, \boldsymbol{\beta}^\dagger_{k'}\right\} = \left(|u_k|^2 + |v_k|^2\right)\delta_{kk'},\\ &\{\boldsymbol{\alpha}_k, \boldsymbol{\alpha}_{k'}\} = \{\boldsymbol{\beta}_k, \boldsymbol{\beta}_{k'}\} = \{\boldsymbol{\alpha}_k, \boldsymbol{\beta}_{k'}\} = \left\{\boldsymbol{\alpha}^\dagger_k, \boldsymbol{\beta}^\dagger_{k'}\right\} = \left\{\boldsymbol{\alpha}_k, \boldsymbol{\beta}^\dagger_{k'}\right\} = 0.\end{aligned}$$

(b) Which condition do u_k and v_k have to satisfy, so that $\boldsymbol{\alpha}_k$ and $\boldsymbol{\beta}_k$ are fermionic annihilators?

(c) Express the four $\mathbf{c}$-operators by the $\boldsymbol{\alpha}$- and $\boldsymbol{\beta}$-operators.