Exercise 12 for Theoretical Solid State Physics in Summer 2023

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1. **Thermodynamics of the superconducting state (3 points)**
   The equilibrium state of a superconducter in a uniform magnetic field is determined by the temperature T and the magnitude of the field H. The thermodynamic identity is written in terms of the Gibbs free energy,

   $$\mathrm{d}G = -S\mathrm{d}T - M\mathrm{d}H + \frac{1}{4\pi}H\mathrm{d}H,$$

   where S is the entropy and M the total magnetization ($M = mV$, where m is the magnetization density). The phase boundary between the superconducting and the normal states in the H-T plane is given by the critical field curve $H_c(T)$ (cf the lecture). Deduce, from the fact that G is continuous across the phase boundary, that

   $$\frac{\mathrm{d}H_c(T)}{\mathrm{d}T} = \frac{S_n - S_s}{M_s - M_n}.$$

2. **Ginzburg-Landau theory (10 points)**
   Ginzburg and Landau postulated the existence of a wave function $\psi(\vec{r})$ describing the phenomenon of superconductivity:

   $$|\psi(\vec{r})|^2 = n_s(\vec{r}),$$

   where $n_s(\vec{r})$ is the density of superconducting particles (mass m^*, charge e^*). For the enthalpy density they made the ansatz:

   $$g_s\left(T,\vec{H}\right) = g_n(T,0) + a|\psi(\vec{r})|^2 + \frac{b}{2}|\psi(\vec{r})|^4 + \frac{\left|\left(\frac{\hbar}{i}\vec{\boldsymbol{\nabla}} - \frac{e^*}{c}\vec{A}(\vec{r})\right)\psi(\vec{r})\right|^2}{2m^*} + \frac{\vec{B}^2}{8\pi} - \frac{\vec{B}\vec{H}}{4\pi}.$$

   $g_n(T,0)$ is the enthalpy density in the normal state. $\vec{B} = \vec{\boldsymbol{\nabla}} \times \vec{A}(\vec{r})$ is the magnetic field in the superconductor. The enthalpy depends on the temperature T and the external magnetic field $\vec{H}$. The condition that the enthalpy $G = \int_V \mathrm{d}^3 r g_s(\vec{r})$ is minimal leads to the thermodynamics of the superconducting state.

(a) First of all consider the case $\vec{A}(\vec{r}) = 0$ and $\psi(\vec{r}) = \text{const.} \neq 0$ and calculate a and b in terms of the particle density $n_s(\vec{r})$ and the critical field H_c.

(b) In order to analyse the effects of surfaces we readmit spatial variations of $\psi(\vec{r})$ and $\vec{A}(\vec{r}) \neq 0$. From the variation of G with respect to $\psi^*(\vec{r})$ and $\vec{A}(\vec{r})$ follows

$$\frac{\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e^*}{c}\vec{A}(\vec{r})\right)^2}{2m^*}\psi(\vec{r}) + a\psi(\vec{r}) + b\left|\psi(\vec{r})\right|^2\psi(\vec{r}) = 0, \tag{1}$$

$$\vec{j}_s := \frac{\hbar e^*}{2im^*}\left(\psi^*(\vec{r})\,\vec{\nabla}\psi(\vec{r}) - \psi(\vec{r})\,\vec{\nabla}\psi^*(\vec{r})\right) - \frac{e^{*2}}{cm^*}\left|\psi(\vec{r})\right|^2\vec{A}(\vec{r}) = \frac{c}{4\pi}\vec{\nabla} \times \left(\vec{B} - \vec{H}\right). \tag{2}$$

Prove these equations.
Hint: The first step is an integration by parts to eliminate terms with $\vec{\nabla}\psi^*(\vec{r})$. Note that surface terms vanish upon local variations. For the proof of the second equation use the identity $\vec{\nabla}\left(\vec{a} \times \vec{b}\right) = \left(\vec{\nabla} \times \vec{a}\right)\vec{b} - \vec{a}\left(\vec{\nabla} \times \vec{b}\right)$ for any $\vec{a}$, $\vec{b}$ and Gauss's theorem.

(c) Consider a superconductor in the half space $x > 0$ and a normal conductor in the other half space $x < 0$. Equation (1) is reduced to a one-dimensional differential equation. Set $f(x) := \frac{\psi(x)}{|\psi(\infty)|}$ and solve the differential equation with the ansatz $f(x) = A\,\mathrm{th}(\alpha(x - x_0))$. Calculate A, α and x_0.