FQM problem Sheet 6 in WS 2022/2023

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1. **Wick's Theorem (part I)** (12)
   We consider a system of non-interacting fermions described by a Hamiltonian operator of the form $\boldsymbol{H} = \Sigma_\alpha \epsilon_\alpha \boldsymbol{c}_\alpha^\dagger \boldsymbol{c}_\alpha$.

   (a) **Show** first the identity

   $$\boldsymbol{c}_\alpha^\dagger e^{-\beta \boldsymbol{H}} = e^{\beta\epsilon_\alpha} e^{-\beta \boldsymbol{H}} \boldsymbol{c}_\alpha^\dagger.$$

   (b) **Calculate** the two-point correlator

   $$\langle \boldsymbol{c}_\alpha^\dagger \boldsymbol{c}_\beta \rangle = \frac{tr\left(e^{-\beta \boldsymbol{H}} \boldsymbol{c}_\alpha^\dagger \boldsymbol{c}_\beta\right)}{tr\left(e^{-\beta \boldsymbol{H}}\right)}$$

   Note: Permute the operator $\boldsymbol{c}_\alpha^\dagger$ to the right using appropriate commutator relations. Then use the cyclicity of the trace and the result from part (a).

   (c) **Calculate**

   $$\langle \psi^\dagger(\vec{x})\psi(0)\rangle \tag{1}$$

   for a system of free particles in a box of size V (finite but large). In this case the Hamiltonian is of the above form with $\alpha \simeq \vec{k}$ and $\epsilon_{\vec{k}} = \hbar^2 k^2/2m - \mu$ where μ is the chemical potential and the $\vec{k}$ are discretely distributed in momentum space. Use the Fourier transform relation of $\psi(\vec{x})$ and $c_{\vec{k}}$ as well as the explicit results for $\langle c_{\vec{k}}^\dagger c_{\vec{q}}\rangle$ derived under (b) for obtaining an integral expression for (1). (Note for large V we have $\sum_k ... = \frac{V}{(2\pi)^3}\int d^3k....$)

   (d) **Carry** out the momentum integral derived under (c) in the limit $T \to 0$ and just in the 1-dimensional case.

2. **Coherent States** (8)
   Let Φ and $\Phi^\dagger$ be bosonic operators ($[\Phi, \Phi^\dagger] = 1$).

   (a) **Find** all eigen-states $|\varphi\rangle$ of Φ
   Note: Make an ansatz of the form $\sum_{n=0}^{\infty} a_n \cdot (\Phi^\dagger)^n |0\rangle$

   (b) What is $\langle x|\varphi\rangle$, if $\Phi = \frac{1}{\sqrt{2}}(k_0\boldsymbol{x} + i\boldsymbol{p}/k_0)$, $k_0 = \sqrt{m\omega}$
   Reminder: The ladder operators of the quantum mechanical harmonic oscillator are bosonic.