FQM problem Sheet 5 in WS 2022/2023

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Submission: 16.11.2022, 12:00 in the P.O. Box Karabin on D.10
Discussion: 16.11.2022, 14:00 – 16:00

1. **Generator of rotations** (9)
   Apply the operator $\exp(-i\varphi \boldsymbol{e}\cdot\boldsymbol{J})$, $\boldsymbol{J} := \boldsymbol{L}+\boldsymbol{S}$ to the plane wave solution
   $$\psi = \begin{pmatrix}\phi\\ \chi\end{pmatrix} e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-Et)}\,,\ \phi = \boldsymbol{v}\in\mathbb{R}^2 \text{ and } \chi = \frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{p_0+m}\boldsymbol{v}$$
   of the Dirac equation.
   Show that the result corresponds to a 'twisted' plane wave, i.e. the transformation $\boldsymbol{v}\mapsto\boldsymbol{v}(\varphi)=\exp(-i\varphi\boldsymbol{e}\cdot\boldsymbol{S})\boldsymbol{v}$ and $\boldsymbol{p}\mapsto\boldsymbol{p}(\varphi)=\boldsymbol{D}(\varphi)\boldsymbol{p}$, where $\boldsymbol{D}(\varphi)$ is the rotation in $\mathbb{R}^3$ around $\boldsymbol{e}$ with angle φ.
   Hint: Use the identity $(\exp(-i\varphi\boldsymbol{e}\cdot\boldsymbol{L})\psi)(\boldsymbol{x}) = \psi(\boldsymbol{D}(\varphi)^{-1}\boldsymbol{x})$.

2. **Quantization of the Dirac field** (12)
   $u(p,s)$ ($v(p,s)$) is the spinor for a solution of the Dirac equation with positive (negative) energy, momentum p and spin s.
   $$(\not{p}-m)u(p,s)=0, \qquad (\not{p}+m)v(p,s)=0$$
   (a) Use the plane wave solutions and the transformation $\psi(p)\to\gamma_5\psi(p)$ to **construct** $u(p,s)$ and $v(p,s)$, where the normalization condition
   $$\bar{u}(p,s)u(p,s')=\delta_{ss'}=-\bar{v}(p,s)v(p,s')$$
   is to be fulfilled.
   (b) **Show** the orthogonality relations
   $$u^\dagger(p,s)u(p,s')=\frac{E}{m}\delta_{ss'}=v^\dagger(p,s)v(p,s')$$
   $$\bar{v}(p,s)u(p,s')=0=v^\dagger(p,s)u(-p,s').$$
   (c) **Show** the completeness relations
   $$\sum_{\pm s}[u_\alpha(p,s)\bar{u}_\beta(p,s)-v_\alpha(p,s)\bar{v}_\beta(p,s)]=\delta_{\alpha\beta}$$
   $$\sum_{\pm s}u_\alpha(p,s)\bar{u}_\beta(p,s)=\left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta}$$
   $$\sum_{\pm s}-v_\alpha(p,s)\bar{v}_\beta(p,s)=\left(\frac{-\not{p}+m}{2m}\right)_{\alpha\beta}$$