FQM problem Sheet 4 in WS 2022/2023

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Discussion: 11.11.2022, 10:00 – 12:00

1. **Gordon Identity** (8)
   Show that the solutions of the Dirac equation in momentum space satisfy the following identity:
   $$\bar{\psi}(p')\gamma^\mu\psi(p) = \bar{\psi}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]\psi(p),$$
   where $q = p' - p$, $\bar{\psi}(p) = \psi^\dagger(p)\gamma^0$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, as well as $(\not{p} - m)\psi(p) = 0$.

2. **Bilinear Covariants** (8)
   Prove the following transformation laws:

   (a)
   $$\bar{\psi}'(\boldsymbol{x}')\psi'(\boldsymbol{x}') = \bar{\psi}(\boldsymbol{x})\psi(\boldsymbol{x}) \qquad \text{(scalar)}$$

   (b)
   $$\bar{\psi}'(\boldsymbol{x}')\gamma_5\psi'(\boldsymbol{x}') = \bar{\psi}(\boldsymbol{x})S^{-1}\gamma_5 S\psi(\boldsymbol{x}) = \det(\Lambda)\bar{\psi}(\boldsymbol{x})\gamma_5\psi(\boldsymbol{x}) \qquad \text{(pseudo-scalar)}$$

   (c)
   $$\bar{\psi}'(\boldsymbol{x}')\gamma^\nu\psi'(\boldsymbol{x}') = \Lambda^\nu{}_\mu\bar{\psi}(\boldsymbol{x})\gamma^\mu\psi(\boldsymbol{x}) \qquad \text{(vector)}$$

   (d)
   $$\bar{\psi}'(\boldsymbol{x}')\gamma_5\gamma^\nu\psi'(\boldsymbol{x}') = \det(\Lambda)\Lambda^\nu{}_\mu\bar{\psi}(\boldsymbol{x})\gamma_5\gamma^\mu\psi(\boldsymbol{x}) \qquad \text{(pseudo-vector)}$$

   (e)
   $$\bar{\psi}'(\boldsymbol{x}')\sigma^{\mu\nu}\psi'(\boldsymbol{x}') = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\bar{\psi}(\boldsymbol{x})\sigma^{\alpha\beta}\psi(\boldsymbol{x}) \qquad \text{(tensor of rank two)}$$

   Here $S = S(\Lambda)$ is the transformation matching the Lorentz transformation Λ in spinor space. [1]
   hint: Use $\psi'(\boldsymbol{x}') = S\psi(\boldsymbol{x})$, and $S^{-1} = \gamma^0 S^\dagger\gamma^0$.

3. **Decoupling of the Lorentz Algebra** (6)
   Show the relations given in the lecture
   $$[e_k^+, e_l^-] = 0 \quad \text{and} \quad [e_1^\pm, e_2^\pm] = ie_3^\pm \qquad \text{(plus cycl. permutations)} \tag{1}$$
   Note that the objects $e_k^\pm$ are defined by
   $$e^\pm_{k\in\{1,2,3\}} = \frac{1}{2}(i \cdot d_k \pm b_k)\,, \tag{2}$$
   and b_k and d_k are the infinitesimal boosts and rotations.

[1] $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, $\bar{\psi}(p) = \psi^\dagger(p)\gamma^0$ and $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$