FQM problem Sheet 3 in WS 2022/2023

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1. **Dirac Algebra** (10)
   Consider the Dirac matrices $\gamma^\mu, \mu = 0, 1, 2, 3$, whose anti-commutators are given by $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. The Dirac matrices are thus the generators of a representation of the Clifford algebra $Cl(3, 1)$.
   Show the following identities, using Feynman's slash notation $\not{a} = \gamma^\mu a_\mu$.

   (a) $\{\not{a}, \not{b}\} = 2a \cdot b, \qquad tr(\not{a}\not{b}) = 4a \cdot b$

   (b) $\gamma_\mu \gamma^\mu = 4, \qquad \gamma_\mu \not{a} \gamma^\mu = -2\not{a}, \qquad \gamma_\mu \not{a}\not{b} \gamma^\mu = 4a \cdot b, \qquad \gamma_\mu \not{a}\not{b}\not{c} \gamma^\mu = -2\not{c}\not{b}\not{a}$

2. **Lorentz Algebra** (12)
   The defining representation of the Lorentz algebra $\mathfrak{o}(3, 1)$ is given by the generators

$$(\boldsymbol{J}^{\mu\nu})^\alpha{}_\beta := g^{\mu\alpha} g^\nu{}_\beta - g^{\nu\alpha} g^\mu{}_\beta \tag{1}$$

   where strictly speaking, only the 6 linearly independent elements with $\mu < \nu$ are meant.

   (a) The connection with the lecture is given through $\boldsymbol{J}^{0k} = b_k$ $(k = 1, 2, 3)$ and $\boldsymbol{J}^{12} = d_3$ plus cyclic permutation. Check this.

   (b) Check the commutator relation

$$[\boldsymbol{J}^{\mu\nu}, \boldsymbol{J}^{\rho\sigma}] = g^{\nu\rho} \boldsymbol{J}^{\mu\sigma} - g^{\mu\rho} \boldsymbol{J}^{\nu\sigma} - g^{\nu\sigma} \boldsymbol{J}^{\mu\rho} + g^{\mu\sigma} \boldsymbol{J}^{\nu\rho} \tag{2}$$

   by use of (1).

   (c) Show that $\mathfrak{J}^{\mu\nu} := i(\boldsymbol{x}^\nu \boldsymbol{p}^\mu - \boldsymbol{x}^\mu \boldsymbol{p}^\nu)$ also satisfy relation (2) upon substituting $\boldsymbol{J} \to \mathfrak{J}$. (This is a representation in the space of functions on $\mathbb{R}^4$.)