FQM problem Sheet 2 in WS 2022/2023

Prof. Dr. Andreas Klümper (kluemper@uni-wuppertal.de D.10.07)
Svyatoslav Karabin (sv.karabin@gmail.com D.10.01)

Submission: 26.10.2022, 12:00 (possibly later) in the P.O. Box Karabin on D.10
Discussion: 28.10.2022, 10:00 – 12:00

1. **Temple's Operator** (9)
   Consider the operator

$$\boldsymbol{\Lambda} := -(2\boldsymbol{L}\cdot\boldsymbol{S}+1) - i\frac{K}{r}\boldsymbol{\alpha}\cdot\boldsymbol{r},$$

   where $r = |\boldsymbol{r}|$, $\alpha^k = \gamma^0\gamma^k$ $(k = 1, 2, 3)$, $\boldsymbol{S} = \frac{\boldsymbol{\sigma}}{2}$ and "$\cdot$" denotes the standard scalar product.

   Show that

$$\boldsymbol{\Lambda}(\boldsymbol{\Lambda}+1) = \boldsymbol{L}^2 - K^2 - \frac{iK}{r}\boldsymbol{\alpha}\cdot\boldsymbol{r}.$$

2. **The Foldy-Wouthuysen Transformation** (12)
   The Hamilton is given by

$$\boldsymbol{H} := \boldsymbol{\alpha}\cdot(\boldsymbol{p} - e\boldsymbol{A}) + \boldsymbol{\gamma}^0 m + e\Phi.$$

   We are looking for a transformation that decouples the Dirac equation $i\frac{\partial}{\partial t}\psi = \boldsymbol{H}\psi$ into two two-component equations. Since decoupling is generally not possible, we consider in the following a canonical transformation, which realizes this in the form of a non-relativistic expansion.

   (a) Carry out the transformation[1] $\Psi' := e^{i\boldsymbol{S}}\psi$ and derive an equation of the form $i\frac{\partial}{\partial t}\psi' = \boldsymbol{H}'\psi'$.

   (b) Expand $\boldsymbol{H}'$ to first order in $\frac{1}{m}$ under the assumption $\boldsymbol{S} = \mathcal{O}(\frac{1}{m})$, i.e. $\boldsymbol{H}' = \boldsymbol{H}'_1 + \mathcal{O}\left(\frac{1}{m^2}\right)$. Use the following identities for this

$$e^{i\boldsymbol{S}}\boldsymbol{H}e^{-i\boldsymbol{S}} = e^{ad_{i\boldsymbol{S}}}\boldsymbol{H} \doteq \boldsymbol{H} + i[\boldsymbol{S},\boldsymbol{H}] + \frac{i^2}{2!}[\boldsymbol{S}[\boldsymbol{S},\boldsymbol{H}]] + \dots \quad ,$$

$$e^{i\boldsymbol{S}}\left(-i\frac{\partial}{\partial t}\right)e^{-i\boldsymbol{S}} = \frac{1-e^{ad_{i\boldsymbol{S}}}}{ad_{\boldsymbol{S}}}\dot{\boldsymbol{S}} \doteq -\dot{\boldsymbol{S}} - \frac{i}{2!}[\boldsymbol{S},\dot{\boldsymbol{S}}] - \frac{i^2}{3!}[\boldsymbol{S}[\boldsymbol{S},\dot{\boldsymbol{S}}]] + \dots \quad .$$

   (c) Think about why $\boldsymbol{H}'$ by choosing $i\boldsymbol{S} := \frac{\gamma^0}{2m}\boldsymbol{\alpha}\cdot(\boldsymbol{p}-\boldsymbol{eA})$ becomes block diagonal in zeroth order. With this choice, calculate $\boldsymbol{H}'_1$. Do certain terms of the first block of the block diagonal part of $\boldsymbol{H}'_1$ look familiar to you?
   Note: Calculate first the commutator $i[\boldsymbol{S},\boldsymbol{H}]$. For all other commutators you only need the terms of order $\frac{1}{m}$!

[1] Note that $\boldsymbol{S}$ can depend on the time.