FQM problem Sheet 14 in WS 2022/2023

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1. **Representation of $\mathfrak{su}(3)$ with maximum weight $2\mu^1$** (10)
   We consider the Lie algebra $\mathfrak{su}(3)$ with the two-dimensional Cartan subalgebra $\mathfrak{h}$ generated by the basis elements h_1 and h_2. As you already know, the roots ($\equiv$ weights of the adjoint representation) are given by $\{\pm\alpha^1, \pm\alpha^2, \pm(\alpha^1+\alpha^2)\}$, where $\alpha^1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\alpha^2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The weights of the defining representation are $\{\mu^1, \mu^2, -(\mu^1+\mu^2)\}$, where $\mu^1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$ and $\mu^2 = \left(-\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$. [Sorry, there is a consistent sign change in μ_2 compared to standard terminology. Following the standard terminology we have: The weights of the defining representation are $\{\mu^1, -\mu^2, \mu^2-\mu^1\}$, where $\mu^1 = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$ and $\mu^2 = \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}\right)$.]

   (a) **Construct** the representation of $\mathfrak{su}(3)$ with maximum weight $2\mu^1$, i.e. $E_{\alpha^1}|2\mu^1\rangle = E_{\alpha^2}|2\mu^1\rangle = 0$ [1].
      i. From the lecture you know $\frac{\alpha\cdot\mu}{\alpha\cdot\alpha} = -\frac{1}{2}(p-q)$. (What is p and q, respectively?)
      ii. Similarly, $[E_\alpha, E_{-\alpha}] \in \mathfrak{h}$. (lecture)
      iii. $[E_\alpha, E_\beta] = 0$ if $\alpha+\beta$ is not a root (and $[E_\alpha, E_\beta] \simeq E_{\alpha+\beta}$ otherwise)

      Hint: Construct new states by application of sutaible E's to the given state. Show the last of the remarks and then use it (among other things) to determine "the p and q" of the new states you construct. Continue the procedure.

2. **Tensor product of two representations** (10)
   We consider the tensor product of two defining representations ([3]) of $\mathfrak{su}(3)$ ($\hat{=}\,[3]\otimes[3]$). This is again a representation of the $\mathfrak{su}(3)$ by componentwise operation:

$$h_\nu = h_{1,\nu} + h_{2,\nu} := h_\nu \otimes 1 + 1 \otimes h_\nu$$
$$E_\alpha = E_{1,\alpha} + E_{2,\alpha} := E_\alpha \otimes 1 + 1 \otimes E_\alpha$$

   Show: $[3]\otimes[3]$ decomposes into a 6-dim. representation, which starts from $|\mu^1\rangle \otimes \mu^1 \hat{=} |2\mu^1\rangle$, and a 3-dim. representation conjugate (i.e. with inverted roots) to [3]: $[\bar{3}]$.

[1] Normalized basis states to weight μ are written as $|\mu\rangle$,