FQM problem Sheet 13 in WS 2022/2023

Prof. Dr. Andreas Klümper (kluemper@uni-wuppertal.de D.10.07)
Svyatoslav Karabin (karabin@uni-wuppertal.de D.10.01)

Submission: 25.01.2023, 12:00 in the P.O. Box Karabin on D.10
Discussion: 25.01.2023, 14:00 – 16:00

1. **Killing form of gl**$(N, \mathbb{R})$ (9)
We consider the group $\mathrm{Gl}(N, \mathbb{R})$ of all real valued regular matrices and the associated Lie algebra $\mathrm{gl}(N, \mathbb{R})$. Prove

   (a) The dimension of the Lie algebra is N^2.

   (b) The Killing form as introduced in the lecture is related to the matrix trace of the defining representation by
$$B(x, y) = 2n\mathrm{Tr}(XY) - 2\mathrm{Tr}(X)\mathrm{Tr}(X) \tag{1}$$
where x, y denote the elements of the abstract Lie algebra and X, Y are their representing $N \times N$ matrices.

   (c) Is the Killing form degenerate?

   Hint: For the solution a calculation is necessary with transparent book keeping. Take as a basis of the Lie algebra all $N \times N$ matrices $E_{i,j}$ which have all matrix elements zero except for the (i, j)-element which is 1. Then calculate the application of $\mathrm{ad}(E_{i,j})\mathrm{ad}(E_{k,l})$ onto an element $E_{g,h}$ by using rules like
$$\mathrm{ad}(E_{k,l}) \cdot E_{g,h} = [E_{k,l}, E_{g,h}] = \delta_{l,g}E_{k,h} - \delta_{h,k}E_{g,l} \tag{2}$$
The result of $\mathrm{ad}(E_{i,j})\mathrm{ad}(E_{k,l}) \cdot E_{g,h}$ will be a linear combination of all $E_{\cdot,\cdot}$. Identify the coefficient of the $E_{g,h}$ term and sum over all g, h.
Alternative hint:
Follow the above idea, but use for $E_{i,j}$ in the calculations $|i\rangle\langle j|$ and $\mathrm{Tr}\big[\big(|i\rangle\langle j|\big)^+ \sum_{k,l} c_{k,l} |k\rangle\langle l|\big] = c_{i,j}$.

2. **Lorentz algebra** (10)
Use the generators of the Lorentz algebra as in section 1.7 of the lecture, namely as
$$\eta \cdot \omega_{\mu,\nu}\,, \qquad \omega_{\mu,\nu} = |\mu\rangle\langle\nu| - |\nu\rangle\langle\mu| \quad \text{and } |\mu\rangle\,, \langle\nu| \text{ are column and row vectors} \tag{3}$$
where η is the diagonal matrix with diagonal elements $(1, -1, -1, -1)$. In total 6 independent elements exist $(0 \leq \mu < \nu \leq 3)$.

   (a) The Killing form for two elements a, b of the algebra can be *calculated* to give
$$B(a, b) = 2\mathrm{Tr}(AB) \tag{4}$$
where A, B are the representing 4×4 matrices as given above.
Hint: The calculation is like for the previous problem on $\mathrm{gl}(N, \mathbb{R})$, but more involved. We still have a useful relation like $\mathrm{Tr}\big[\omega^+_{\mu\nu} \sum_{0\leq\alpha<\beta\leq3} c_{\alpha\beta}\omega_{\alpha\beta}\big] = 2c_{\mu\nu}$. The $\omega_{\mu\nu}$ however, do not factor "into left and right", they are sums of two such elements.

   (b) If you do not get the calculation done, check the result by use of the explicit adjoint representations of the elements b_3 and d_3 of the lecture.

   (c) Is the Killing form degenerate? Is it positive (or negative) definite? What might be the consequences?