FQM problem Sheet 12 in WS 2022/2023

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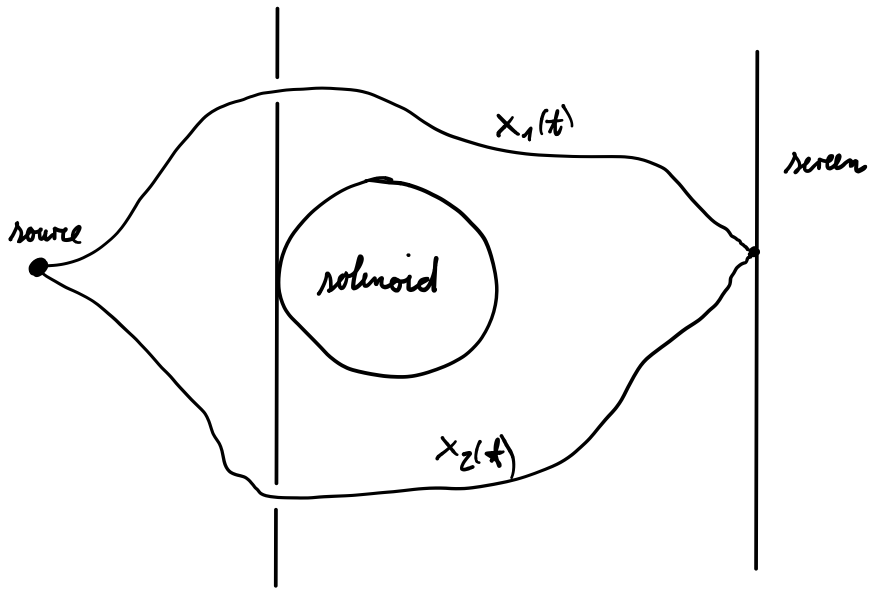
1. **One particle Green's function** (12)
We consider a system of free bosons with Hamiltonian $\mathcal{H} := \int d^3r\, \Phi^+(r)(-\frac{1}{2m}\Delta)\Phi(r)$ bilinear in the fields.

   (a) Assuming a finite volume V in space and an imaginary time interval $[0, \beta]$ with periodic boundary conditions, the action
$$\int_0^\beta d\tau \int_V d^3r \left[\bar{\varphi}(r,\tau)\left\{\frac{d}{d\tau} - \frac{1}{2m}\Delta\right\}\varphi(r,\tau)\right] \tag{1}$$
can be written in Fourier representation as
$$\sum_{p,z} \bar{\varphi}_{p,z} K_{p,z} \varphi_{p,z}. \tag{2}$$
Calculate $K_{p,z}$.
Note: $\varphi(r,\tau) = \frac{1}{\sqrt{V\beta}}\sum_{p,z} e^{ip\cdot r - z\tau}\varphi_{p,z}$ and $\varphi_{p,z} = \frac{1}{\sqrt{V\beta}}\int d^3r d\tau\, e^{-ip\cdot r + z\tau}\varphi(r,\tau)$, $z = \frac{2\pi \mathrm{i}}{\beta}n$. Use the lecture notes from (3.90) to (3.95) (to be presented on Friday, 13.01.2023).

   (b) **Calculate** K^{-1} in space-time representation, namely the object in brackets in (3.95). Leave the momentum summation as it is, but do the sum over $z = \frac{2\pi\mathrm{i}}{\beta}n$ explicitly. For this goal, use the contour integral representations
$$\sum_z \frac{1}{\epsilon - z}e^{-z\tau} = \frac{1}{2\pi\mathrm{i}}\int_I dz \frac{\beta}{1 - e^{-\beta z}}\frac{1}{\epsilon - z}e^{-z\tau} = \frac{1}{2\pi\mathrm{i}}\int_I dz \frac{\beta}{e^{\beta z} - 1}\frac{1}{\epsilon - z}e^{-z\tau} \tag{3}$$
where I is a narrow contour in anti-clockwise manner around the entire imaginary axis, and τ is an abbreviation for $\tau' - \tau \in [-\beta, \beta]$. Depending on the sign of this value, the first or the second integral representation allows for a deformation of the contour (without change of the value of the integral) to a new contour consisting of two parts: one is around the negative part of the real axis (excluding 0), the other is around the positive part of the real axis (excluding 0), both in clockwise manner. These integrals can be done easily, in fact only one surrounds a single pole.

2. **Aharonov-Bohm effect in path integral formalism** (9)
We consider the double-slit experiment with electrons. A magnetic coil is placed directly behind the double slit. This coil generates a magnetic field inside the coil and a negligible field outside. Further, it is to be assumed that contributions from paths where the electron crosses the area of the coil can be neglected. Similarly, it can be assumed that only paths where the electron crosses exactly one of the gaps contribute to the path integral. Let the magnetic field of the coil be orthogonal to the image plane. Let the Lagrangian for the motion of the electron be $\mathcal{L} = \mathcal{L}_0 - e\dot{\mathbf{x}}\cdot\mathbf{A}$, where $\mathbf{A}$ is the vector potential of the magnetic field and $\mathcal{L}_0$ is the Lagrangian for describing the motion of the electron in the absence of the magnetic field.

   (a) **Show** that contributions of arbitrary paths through the first and second slits, respectively, are of the form
$$K_1 = \exp\left(-ie\int_{x_1} d\mathbf{x}\cdot\mathbf{A}\right)K_{1,0} \tag{4}$$
$$K_2 = \exp\left(-ie\int_{x_2} d\mathbf{x}\cdot\mathbf{A}\right)K_{2,0} \tag{5}$$
where $K_{1/2,0}$ are the contributions to the path integral without magnetic field.
Explain why these expressions do not depend on the shape of the paths $x_1(t)$ and $x_2(t)$.



(b) **Show** that the total amplitude $K = K_1 + K_2$ can be written as

$$K = \exp\left(-ie \int_{x_1} d\mathbf{x} \cdot \mathbf{A}\right) (K_{1,0} + \exp(-i\phi) K_{2,0}) \tag{6}$$

where ϕ depends solely on the magnetic field inside the coil. **Relate** ϕ to the magnetic flux through the solenoid.

(c) **Explain/discuss** that the presence of the magnetic field can be determined from the interference pattern on the screen, even if all paths in the path integral pass exclusively through a region with zero magnetic field.