# FQM problem Sheet 1 in WS 2022/2023

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1. **KG scattering at a potential threshold** (10)
   The Klein-Gordon equation in $1+1$ D (where the first number stands for the dimension of space, the second for that of time) for a particle in an electrical potential $V(x,t)$ reads:
   $$(i\partial_t - V(x,t))^2\psi(x,t) = (-\partial_x^2 + m^2)\psi(x,t)$$
   (**Remark**: $\hbar \equiv c \equiv 1$)
   Consider the time independent potential
   $$V(x) = \begin{cases} 0 & , \quad x \leq 0 \\ V & , \quad x > 0 \end{cases}.$$
   Apply the ansatz $\psi(x,t) = \exp(-itE)\psi(x)$ for deriving the stationary KG equation and solve it for a plane wave incident from the left (from $-\infty$) of energy $E = \sqrt{p^2 + m^2}$:
   $$\psi(x) = \begin{cases} e^{ipx} + Re^{-ipx} & , \quad x \leq 0 \\ Te^{iqx} & , \quad x > 0 \end{cases}.$$
   It makes sense to distinguish the cases
   (a) $V < E - m$,
   (b) $E - m < V < E + m$,
   (c) $E + m < V$.

   Determine $R$ and $T$, as well as the ratio of the reflected and transmitted current density $j_t$ and $j_r$ to the incident current density $j_{in}$. How can the result be interpreted physically?

   **Remark**: Pay particular attention to the sign of the momentum $q$, which can be determined by the physical requirement of the positivity of the group speed $v_g = \partial_q E$.
   **Reminder**: The current density of a wave function $\psi$ in one dimension is $j = \frac{1}{2mi}(\psi^*\partial_x\psi - \psi\partial_x\psi^*)$

2. **KG equation in a Coulomb potential** (10)
   The aim of this task is to calculate relativistic corrections to the energy levels of bound states with the help of the Klein-Gordon equation. Since the KG equation determines the kinematics of scalar fields (spin 0), for example bound states of the pion $\pi^-$ in the Coulomb potential of an atomic nucleus can be calculated.

   (a) Write the KG equation ($3+1$ D) in analogy to exercise 1 in stationary form with the time-independent potential
   $$V(\mathbf{x}) = \frac{-K}{r}\ ,\ r = |\mathbf{x}|$$
   (b) Transform the equation into spherical coordinates and, as in QM I, make the separation ansatz[1] $\Psi(\mathbf{x}) = \psi(r)\chi(\theta,\varphi)$. (**Reminder**: $r^2\mathbf{\Delta} = r\partial_r^2 r - \boldsymbol{L}^2$)
   (c) Compare terms of the same order in $r$ with those of the nonrelativistic Coulomb problems. Do you notice something? Can you give the energy levels of the relativistic problem?

   **Reminder**: The energy levels of the non-relalativistic Coulomb problems are $E = \frac{-mK^2}{2(n_r+l+1)^2}$, where $n_r$ counts the zeros of the radial wave function.
   **Note**: The Relationship $l(l+1) + \frac{1}{4} = (l+\frac{1}{2})^2$ could help.

[1] Let $f$, $g$, $k$ and $l$ be any $\mathbb{C}$-valued functions such that $f(x)g(y) = k(x)l(y)$ holds. Then $f(x) = ck(x)$ and $l(y) = cg(y)$ with a constant $c \in \mathbb{C}$, as long as one excludes trivial solutions. In our case we get $c = l(l+1)$. (Why?)