January 17, 2025

Ex 13: Bethe Ansatz equations of the Bose gas in the strong coupling limit

In the lecture we have obtained the logarithmic Bethe Ansatz equations of the Bose gas with contact interactions,

$$\frac{k_{I_\ell}}{2\pi} + \frac{1}{2\pi L}\sum_{j=1}^{N} \theta(k_{I_\ell} - I_j) + \frac{N+1}{2L} = \frac{I_\ell}{L}\,, \quad \ell = 1, \ldots, N\,, \tag{1}$$

where L is the length of the system, N the number of particles and

$$\theta(k) = 2\,\mathrm{arctg}(k/c)\,.$$

The constant $c > 0$ is the coupling constant that determines the strength of the repulsion between the Bosons.

$\{I_\ell\}_{\ell=1}^N$ is a set of integers that parameterizes the solutions $\{k_{I_\ell}\}_{\ell=1}^N$ of the Bethe Ansatz equations (1). All physical observables pertaining to the model can be expressed in terms of the k_{I_ℓ}. E.g., the energy of the Bethe eigenstate with rapidities $\{k_{I_\ell}\}_{\ell=1}^N$ is

$$E = \sum_{j=1}^{N} k_{I_j}^2\,.$$

(i) Solve the Bethe Ansatz equations (1) perturbatively in the strong coupling limit up to the order $1/c$.

(ii) Obtain the corresponding energy up the order $1/c$.

(iii) Consider the ground state for fixed N, $I_\ell = \ell$, $\ell = 1, \ldots, N$. What is the explicit first order correction in $1/c$ to the ground state energy for fixed N and in the thermodynamic limit $N, L \to \infty$ for fixed $D = N/L$?

(6 points)

Ex 14: Yang-Yang thermodynamics for strong and weak coupling

Obtain the explicit solutions of the Yang-Yang non-linear integral equation

$$\varepsilon(k) = k^2 - h - T\int_{\mathbb{R}} \frac{\mathrm{d}q}{\pi}\frac{c}{(k-q)^2 + c^2}\ln\bigl(1 + \mathrm{e}^{-\frac{\varepsilon(q)}{T}}\bigr)\,,$$

in the limits $c \to \infty$ and $c \to 0$. How does the corresponding grand canonical potential look like and what ist the physical interpretation of the two limiting cases?

(4 points)