December 28, 2024

Ex 12: Bethe states as sl_2 highest-weight states

In the lecture we have considered the L-matrix

$$L(\lambda) = -\mathrm{i}\lambda + \sigma^\alpha \otimes s^\alpha ,$$

where the σ^α, $\alpha = x, y, z$, are the Pauli matrices and the s^α form a basis of sl_2 in such a way that

$$[s^\alpha, s^\beta] = \mathrm{i}\varepsilon^{\alpha\beta\gamma} s^\gamma .$$

The corresponding monodromy matrix associated with an N-site chain is

$$T(\lambda) = L_N(\lambda) \dots L_1(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} .$$

Denote the operators of the total spin of this chain by

$$S^\alpha = \sum_{n=1}^{N} s_n^\alpha .$$

In the lecture we have shown that the monodromy matrix safies the sl_2-invariance condition

$$\left[T(\lambda), \tfrac{1}{2}\sigma^\alpha \otimes \mathrm{id} + I_2 \otimes S^\alpha\right] = 0$$

and that this condition implies $[\operatorname{tr} T(\lambda), S^\alpha] = 0$.

(i) Let $S^+ = S^x + \mathrm{i}S^y$. Show that

$$[S^+, B(\lambda)] = A(\lambda) - D(\lambda) .$$

(ii) Let $|0\rangle$ be a the pseudo vacuum for a spin-s highest weight representation, such that $(s_n^x + \mathrm{i}s_n^y)|0\rangle = 0$, $s_n^z|0\rangle = s|0\rangle$ for $n = 1, \dots, N$. Let $a(\lambda)$, $d(\lambda)$ be the pseudo vacuum eigenvalue functions associated with $A(\lambda)$, $D(\lambda)$, respectively.
Use the formulae for the action of $A(\lambda)$ and $D(\lambda)$ on products of operators $B(\lambda_j)$ from the lecture in order to show that the action of S^+ on an off-shell Bethe state is

$$S^+ B(\lambda_M) \dots B(\lambda_1)|0\rangle = \mathrm{i} \sum_{j=1}^{M} \left[\prod_{\substack{k=1 \\ k \neq j}} B(\lambda_k)\right] |0\rangle \frac{\gamma(\lambda_j|\{\lambda\})}{Q(\lambda_j|\{\lambda\}_j)} ,$$

where

$$\gamma(\mu|\{\lambda\}) = a(\mu)Q(\mu - \mathrm{i}|\{\lambda\}) + d(\mu)Q(\mu + \mathrm{i}|\{\lambda\})$$

and where we have employed the definitions of the Q-functions from the lecture.
Thus, $B(\lambda_M) \dots B(\lambda_1)|0\rangle$ is an sl_2 highest-weight state, if the Bethe Ansatz equations

$$\gamma(\lambda_j|\{\lambda\}) = 0 , \quad j = 1, \dots, N$$

are satisfied.

(iii) Show that an on-shell Bethe vector with $M > N/2$ is equal to zero.

(8 points)