December 3, 2024

Ex 10: On the algebraic Bethe Ansatz for the gl_2-invariant R-matrix

In the lecture we have constructed the algebraic Bethe ansatz solution for the gl_2-invariant R-matrix. An essential step in the construction was the derivation of the commutation relation

$$\begin{aligned} A(\lambda)B(\lambda_1)\dots B(\lambda_M) = {} & \Big[\prod_{j=1}^{M} B(\lambda_j)\Big] A(\lambda) \prod_{j=1}^{M} \frac{1}{b(\lambda_j-\lambda)} \\ & - \sum_{j=1}^{M} \frac{c(\lambda_j-\lambda)}{b(\lambda_j-\lambda)} \Big[B(\lambda) \prod_{\substack{k=1\\k\neq j}}^{M} B(\lambda_k)\Big] A(\lambda_j) \prod_{\substack{k=1\\k\neq j}}^{M} \frac{1}{b(\lambda_k-\lambda_j)}, \end{aligned} \tag{1}$$

where

$$b(\lambda) = \frac{\lambda}{\lambda+\mathrm{i}}, \quad c(\lambda) = \frac{\mathrm{i}}{\lambda+\mathrm{i}}$$

and $A(\lambda)$ and $B(\mu)$ satisfy the Yang-Baxter algebra relations

$$A(\lambda)B(\mu) = \frac{1}{b(\mu-\lambda)} B(\mu)A(\lambda) - \frac{c(\mu-\lambda)}{b(\mu-\lambda)} B(\lambda)A(\mu),$$

$$B(\lambda)B(\mu) = B(\mu)B(\lambda).$$

Prove (1) by induction over M.

Hint: The proof relies on a nontrivial identity for rational functions that has to be guessed and then proved by means of Liouville's theorem.

(6 points)

Ex 11: Creation of a magnon

The Heisenberg chain is associated with the L-matrix

$$L(\lambda) = \begin{pmatrix} -\mathrm{i}\lambda + \frac{1}{2}\sigma^z & \sigma^- \\ \sigma^+ & -\mathrm{i}\lambda - \frac{1}{2}\sigma^z \end{pmatrix}.$$

Consider an N-site chain. Then the corresponding monodromy matrix is

$$\begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = L_N(\lambda)\dots L_1(\lambda).$$

Calculate the action of $B(\lambda)$ on the fully polarized state

$$|0\rangle = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^{\otimes N}.$$

Can you identify a plane wave? What is the interpretation of your result?

(4 points)