November 25, 2024

Ex 8: Momentum

Let $\Psi(x)$ and $\Psi^+(y)$ be conjugate fields with canonical commutation relations

$$[\Psi(x), \Psi^+(y)] = \delta(x-y)\,, \quad [\Psi(x), \Psi(y)] = [\Psi^+(x), \Psi^+(y)] = 0\,.$$

Let

$$P = \frac{\mathrm{i}}{2}\int_{-\infty}^{\infty} \mathrm{d}x\, \big(\Psi^+(x)\partial_x\Psi(x) - (\partial_x\Psi^+(x))\Psi(x)\big)\,.$$

(i) Calculate the commutators $[P, \Psi(x)]$ and $[P, \Psi^+(x)]$. What is the meaning of the operator P?

(ii) For $a \in \mathbb{R}$ calculate

$$\mathrm{e}^{-\mathrm{i}aP}\, \Psi(x)\, \mathrm{e}^{\mathrm{i}aP}\,.$$

(iii) Show that the Hamiltonian

$$H = \int_{-\infty}^{\infty} \mathrm{d}x\, \{(\partial_x\Psi(x)^+)\partial_x\Psi(x) + c\Psi^+(x)\Psi^+(x)\Psi(x)\Psi(x)\}\,,$$

where c is a real coupling constant, commutes with P.

(4 points)

Ex 9: Second factorization of the lattice Bose gas

In the lecture we considered the integrable structure of a lattice Bose gas with contact interaction. It is a model of N Bose fields Ψ_n, Ψ_m^+, $m, n = 1, \dots, N$, with commutation relations

$$[\Psi_n, \Psi_m^+] = \frac{\delta_{nm}}{\Delta}\,, \quad [\Psi_n, \Psi_m] = [\Psi_n^+, \Psi_m^+] = 0\,.$$

The positive parameter Δ is the lattice spacing. We shall further assume that N is even.

An integrable lattice model with local Hamiltonian can be constructed from the L-matrix

$$L_n(\lambda) = \begin{pmatrix} \mathrm{i}\lambda + \alpha_n + \Delta\Psi_n^+\Psi_n & \sqrt{\Delta}\Psi_n^+\, g(2\alpha_n + \Delta\Psi_n^+\Psi_n) \\ f(2\alpha_n + \Delta\Psi_n^+\Psi_n)\sqrt{\Delta}\Psi_n & -\mathrm{i}\lambda + \alpha_n + \Delta\Psi_n^+\Psi_n \end{pmatrix}\,,$$

where, for $\alpha \in \mathbb{R}$,

$$\alpha_n = \alpha + \frac{1}{2}\big(1 + (-1)^n\big)$$

and where the functions f and g satisfy

$$f(2\alpha_n + \Delta\Psi_n^+\Psi_n)g(2\alpha_n + \Delta\Psi_n^+\Psi_n) = 2\alpha_n + \Delta\Psi_n^+\Psi_n\,.$$

(i) Let

$$t(\lambda) = \mathrm{tr}\big\{L_N(\lambda)\dots L_1(\lambda)\big\}$$

be the transfer matrix of the model. According to the lecture $t(-\mathrm{i}\alpha)$ factorized into a product of local factors. Obtain the factorized form explicitly.

(ii) Calculate the logarithmic derivative $\mathrm{i}\partial_\lambda \ln\big(t(\lambda)\big)\big|_{\lambda=-\mathrm{i}\alpha}$ explicitly as a sum over local terms. What is the relation of these terms to the corresponding expression obtained at the other factorization point $\lambda = \mathrm{i}\alpha$ in the lecture?

(8 points)