November 10, 2024

Ex 5: The nonlinear Schrödinger equation

Consider the classical field theory for conjugate fields Ψ and Ψ^* with canonical Poisson brackets,

$$\{\Psi(x), \Psi^*(y)\} = -\mathrm{i}\delta(x-y)\,, \quad \{\Psi(x), \Psi(y)\} = \{\Psi^*(x), \Psi^*(y)\} = 0\,,$$

defined by the Hamiltonian

$$H = \int_{-\infty}^{\infty} \mathrm{d}x\, \{(\partial_x \Psi(x)^*)\partial_x \Psi(x) + c\Psi^*(x)\Psi^*(x)\Psi(x)\Psi(x)\}\,,$$

where c is a real coupling constant.

Calculate the right hand side of the equations of motion

$$\partial_t \Psi = \{\Psi, H\}\,.$$

(4 points)

Ex 6: Holstein-Primakov for classical fields

Consider two canonically conjugate fields Ψ, Ψ^* with Poisson brackets

$$\{\Psi, \Psi^*\} = -\mathrm{i}\,, \quad \{\Psi, \Psi\} = \{\Psi^*, \Psi^*\} = 0\,.$$

Let

$$s^+ = \mathrm{i}f\Psi\,, \quad s^- = -\mathrm{i}\Psi^* g\,, \quad s^z = -\mathrm{i}\alpha - \mathrm{i}\Psi^*\Psi\,,$$

where $\alpha \in \mathbb{C}$ and f and g are functions of $\Psi^*\Psi$.

(i) Show that

$$\{s^+, s^-\} = -\mathrm{i}fg - \mathrm{i}(fg)'\Psi^*\Psi\,,$$

where the prime denotes the derivative.

(ii) Calculate $\{s^z, s^\pm\}$ and give a sufficient condition on f, g for s^z, $s^\pm$ to be generators of a representation of sl_2.

(4 points)

Ex 7: Classical Yang-Baxter algebra for spins

Consider the classical Yang-Baxter algebra in the form

$$\{L_1(\lambda), L_2(\mu)\} = [L_1(\lambda) + L_2(\mu), r_{12}(\lambda - \mu)]\,, \tag{1}$$

where

$$r(\lambda) = \frac{P}{\lambda}$$

and P is the transpoition matrix on $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Show that

$$L(\lambda) = I_2 + \frac{1}{\lambda}\begin{pmatrix} s^z & s^- \\ s^+ & -s^z \end{pmatrix}\,,$$

where s^z, $s^\pm$ satisfy the sl_2 relations

$$\{s^z, s^\pm\} = \pm s^\pm\,, \quad \{s^+, s^-\} = 2s^z\,,$$

is a representation of (1).

(4 points)