October 22, 2024

Ex 1: Unitarity of $\boldsymbol{R}$-matrices

Let $R : \mathbb{C}^2 \mapsto \operatorname{End} \mathbb{C}^d \otimes \mathbb{C}^d$ be a solution of the Yang-Baxter equation,

$$R_{12}(\lambda, \mu) R_{13}(\lambda, \nu) R_{23}(\mu, \nu) = R_{23}(\mu, \nu) R_{13}(\lambda, \nu) R_{12}(\lambda, \mu)\,,$$

which is holomorphic in a vicinity of $(0,0)$ in $\mathbb{C}^2$ and regular in the sense that

$$R(\lambda, \lambda) = P$$

where P is the transposition on $\mathbb{C}^d \otimes \mathbb{C}^d$.

Show that the following lemma holds true.

Lemma. *Under the above conditions there exists a function $g : \mathbb{C}^2 \mapsto \mathbb{C}$ which is differentiable in a neighbourhood of $(0,0)$, has the properties $g(0,0) = 1$, $g(\lambda, \mu) = g(\mu, \lambda)$, and unitarizes R, meaning that in a neighbourhood of $(0,0)$*

$$\frac{R_{12}(\lambda, \mu)}{g(\lambda, \mu)} \frac{R_{21}(\mu, \lambda)}{g(\mu, \lambda)} = \mathrm{id}\ .$$

(8 points)

Ex 2: First higher conserved charge and heat current

Let $R(\lambda, \mu)$ be a regular and unitary R-matrix and

$$t(\lambda) = \operatorname{tr}_a \bigl\{ R_{aL}(\lambda, 0) \dots R_{a1}(\lambda, 0) \bigr\}$$

the associated homogeneous transfer matrix. We have seen in the lecture that

$$\tau(\lambda) = \ln(t^{-1}(0) t(\lambda))$$

is a generating function of conserved charges, meaning that

$$\tau(\lambda) = \lambda I_1 + \lambda^2 I_2 + \dots$$

implies

$$[I_j, I_k] = 0\,.$$

(i) We usually write $H = I_1$ and call this quantity (possibly after changing its normalization and adapting some parameters) 'the Hamiltonian'. Express the Hamiltonian in terms of $\check{R} = PR$.

(ii) Calculate the second conserved charge I_2 in terms of $\check{R} = PR$.

(iii) How does the expression simplify if R is of difference form, $R(\lambda, \mu) = R(\lambda - \mu)$?

(iv) Let $h = \check{R}'(0)$ for an R matrix of difference form. Using the Heisenberg equation of motion, obtain $\dot{h}_{j-1,j}$. How is this quantity related to the second conserved charge and what is thus the physical interpretation of I_2, if R is of difference form?

(10 points)