Homework sheet for lectures of F. Göhmann — Izmir 2014

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Exercise 1 Permuting lattice sites

In the lecture we introduced the canonical basis $\{e^\beta_\alpha | \alpha, \beta = 1, \dots, d\}$ of the space of endomorphisms of the vector space $\mathbb{C}^d$ and its embedding $\{e_j{}^\beta_\alpha | \alpha, \beta = 1, \dots, d; j = 1, \dots, L\}$ into $\mathrm{End}\bigl(\mathbb{C}^d\bigr)^{\otimes L}$. The e^β_α are represented by matrices containing a single non-zero entry 1 in αth row and βth column, i.e. they satisfy the multiplication rule

$$e^\beta_\alpha e^\delta_\gamma = \delta^\beta_\gamma e^\delta_\alpha \, .$$

We claimed that the operators

$$P_{jk} = e_j{}^\beta_\alpha e_k{}^\alpha_\beta$$

(for $j \neq k$) induce the action of the symmetric group on $\mathrm{End}\bigl(\mathbb{C}^d\bigr)^{\otimes L}$. Prove this claim!

For this purpose show that

$$\begin{aligned} & P_{jk} P_{kl} = P_{jl} P_{jk} = P_{kl} P_{jl} \quad \text{if } j \neq k \neq l \neq j \\ & P_{jk}^2 = 1 \\ & [P_{jk}, P_{lm}] = 0 \quad \text{if } j, k \neq l, m \, . \end{aligned}$$

Exercise 2 QTM at infinite temperature

In the lecture we defined a quantum transfer matrix $t(\lambda)$ for every regular, unitary R-matrix $R(\lambda, \mu) \in \mathrm{End}\bigl(\mathbb{C}^d \otimes \mathbb{C}^d\bigr)$ by

$$t(\lambda) = \mathrm{tr}_a \, R_{a,N}(\lambda, \beta/N) R^{t_1}_{N-1,a}(-\beta/N, \lambda) \dots R_{a,2}(\lambda, \beta/N) R^{t_1}_{1,a}(-\beta/N, \lambda) \, .$$

Here N is the Trotter number, β is the inverse temperature and the superscript t_1 denotes the transpose with respect to the first factor in the tensor product R is acting on.

Using the regularity calculate

$$t(0)\Big|_{\beta=0} = t_0$$

and show that t_0/d is a one-dimensional projector. What does this mean for the spectrum of the quantum transfer matrix at infinite temperature?

Exercise 3 High-temperature expansion of the free energy of the XXZ chain

In the lecture we calculated the free energy per lattice site for the XXZ chain in the thermodynamic limit. In the critical regime for $0 \le \Delta < 1$ we obtained

$$f(T, h) = -\frac{h}{2} - T \int_{\mathcal{C}} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}} \mathrm{e}(\lambda) \ln\bigl(1 + \mathfrak{a}(\lambda)\bigr) \, ,$$

where T is the temperature, h is the magnetic field in z-direction and

$$\mathrm{e}(\lambda) = \mathrm{cth}(\lambda) - \mathrm{cth}(\lambda + \eta)$$

is the bare energy. The integration contour $\mathcal{C}$ encircles the real axis and cuts the imaginary axis infinitesimally below $-\eta = \mathrm{i}\gamma$, $\gamma > 0$, and infinitesimally above η. The auxiliary function $\mathfrak{a}$ is the solution of the non-linear integral equation

$$\ln \mathfrak{a}(\lambda) = -\frac{h}{T} - \beta \mathrm{e}(\lambda) - \int_{\mathcal{C}} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} K(\lambda - \mu) \ln\bigl(1 + \mathfrak{a}(\mu)\bigr)$$

with integration kernel

$$K(\lambda) = \mathrm{cth}(\lambda - \eta) - \mathrm{cth}(\lambda + \eta) \, .$$

In order to refer to a common normalization of the Hamiltonian we set

$$\beta = \frac{2J \, \mathrm{sh}(\eta)}{T} \, .$$

Calculate the first terms of the high-temperature expansion of f from this formulation!

Remark: Using a computer algebra program one can rather easily obtain the first 30 terms of the high-temperature expansion.